Instability of Nonuniform Density Free Shear Layers with a Wake Profile

M. M. Koochesfahani* and C. E. Frieler†
California Institute of Technology, Pasadena, California

The linear spatial instability characteristics of both uniform and nonuniform density plane mixing layers were investigated taking into account the wake component of the initial velocity profile. Two unstable modes were found. In the shear layer mode, the growth of the unstable disturbance leads to the usual Kelvin–Helmholtz rollup pattern, whereas in the wake mode, rollup patterns resemble those in wake flows. It was found that the shear layer mode dominates the wake mode when the density is uniform across the layer. The wake mode, however, can become comparable or even stronger than the shear layer mode if the density of the low-speed stream is larger than that of the high-speed stream. Experimental evidence in support of these findings is provided.

Introduction

THE inviscid linear instability of two-dimensional two-stream plane mixing layers has been studied extensively in the past. In the case of uniform density, Michalke[1] investigated the single-stream shear layer, while the effect of the velocity ratio in two-stream mixing layers was considered by Monkewitz and Huerre.[2] Maslowe and Kelly[3] studied stratified (nonuniform density) shear layers and showed the density variations can be destabilizing. In all these studies, the mean velocity profile has been assumed to be monotonically increasing from the value on the low-speed stream to that on the high-speed stream and usually the hyperbolic tangent form is used. It should be noted, however, that under experimental conditions the initial mean velocity profile almost always has a wake component due to the boundary layers on the two sides of the splitter plate. The effect of the wake component has only recently come into consideration with the investigations of Miau[4] and Zhang et al.[5] for the uniform density case.

Recent experimental results[6] have indicated that the initial behavior (near the splitter plate) of a nonuniform density shear layer may be quite different from that of a uniform density layer, leading to the well-known Kelvin–Helmholtz rollup pattern. The flow appears to be more wake-like and has much lower growth and entrainment rates. It was thought that the existence of a wake component in the initial velocity profile of the shear layer might be responsible for this peculiar behavior. The purpose of the present work is to understand this behavior by studying the instability characteristics of both uniform and nonuniform density plane shear layers whose initial velocity profiles include a wake component. The inviscid, linear, parallel-flow stability analysis of spatially growing disturbances is utilized to numerically calculate the range of unstable frequencies and wave numbers. The flow patterns resulting from the amplification of the instability are examined by calculating the streaklines and are compared with the experimental flow-visualization pictures.

In this study, the effects of gravity (i.e., buoyancy) in the case of nonuniform density shear layer have been ignored. The results of Koop and Browand[7] have shown that for small enough values of the Richardson number, the effects of buoyancy on the shear layer can be safely neglected. The Richardson number is defined by

$$Ri = \frac{\Delta\rho g \delta}{\bar{\rho}(\Delta U)^2}$$

where $\Delta\rho = \rho_1 - \rho_2$ is the density difference across the layer, $\bar{\rho} = (\rho_1 + \rho_2)/2$ the average density, g the gravitational constant, δ the shear layer local vorticity thickness, and $\Delta U = U_1 - U_2$ the velocity difference across the layer. Strictly speaking, therefore, the present results apply to shear layers for which the Richardson number is very small.

Formulation of the Problem

We consider the general case of a two-stream plane shear layer with U_1, ρ_1 as the freestream velocity and density on the high-speed stream and U_2, ρ_2 as the corresponding quantities on the low-speed side of the layer. All of the quantities used here are normalized with the average velocity $(U_1 + U_2)/2$, average density $(\rho_1 + \rho_2)/2$, and local layer thickness δ as the length scale. Since we are not aware of any exact solutions for the initial evolution of nonuniform density mixing layers with a wake component, we assume that the mean velocity and density profiles have the following forms. The mean velocity profile is composed of the usual hyperbolic tangent profile plus a wake component (due to the splitter plate) represented by a Gaussian distribution and has the form

$$U(y) = 1 + \lambda_u \tanh(y) - We^{-\ln(2)y^2} \tag{1}$$

where $\lambda_u = (U_1 - U_2)/(U_1 + U_2)$ and W is the normalized wake deficit. The mean density profile has a hyperbolic tangent profile and is given by

$$\rho(y) = 1 + \lambda_\rho \tanh\left[\frac{(y - y_0)}{\sigma}\right] \tag{2}$$

where $\lambda_\rho = (\rho_1 - \rho_2)/(\rho_1 + \rho_2)$, and y_0 and σ adjust the lateral

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*Post-Doctoral Research Fellow; currently, Assistant Professor, Department of Mechanical Engineering, Michigan State University, East Lansing, MI. Member AIAA.

†Graduate Student, Aeronautics.

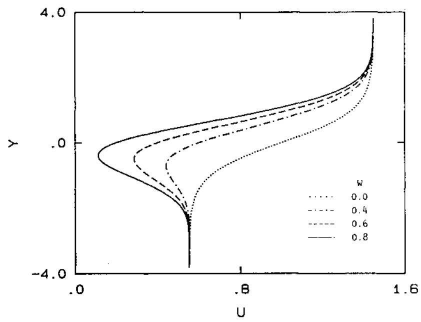


Fig. 1 Mean velocity profiles for different values of W, $\lambda_u = 0.45$ ($U_2/U_1 \approx 0.38$).

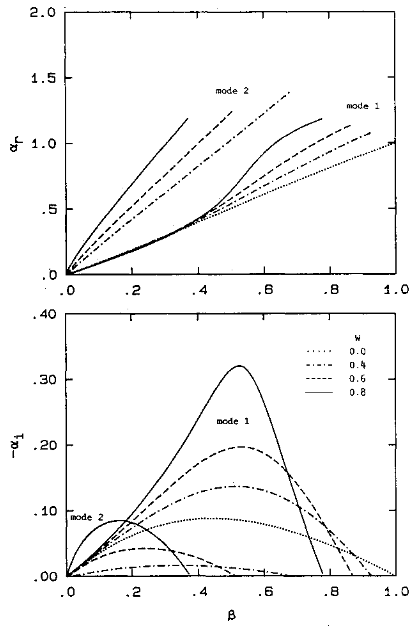


Fig. 2 Instability characteristics for different values of W, $\lambda_u = 0.45$ ($U_2/U_1 \approx 0.38$), uniform density.

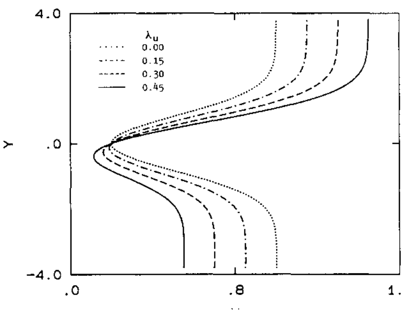


Fig. 3 Mean velocity profiles for different values of λ_u, $W = 0.8$.

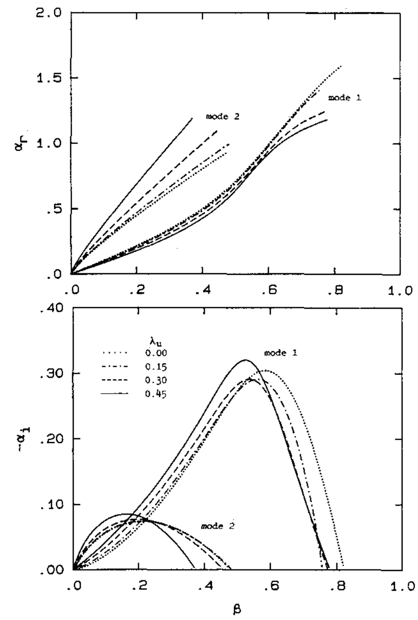


Fig. 4 Instability characteristics for different values of λ_u, $W = 0.8$, uniform density.

position and thickness of the density profile relative to the velocity profile.

The disturbance stream function is written in the form

$$\psi = \phi(y)e^{i(\alpha x - \beta t)} \tag{3}$$

where $\alpha = \alpha_r + i\alpha_i$ is the complex nondimensional wave number and β is the nondimensional frequency, which is taken as pure real for the present spatial calculations. In the case of two-dimensional incompressible flow with negligible buoyancy effects (i.e., gravity is ignored), it can be shown that the disturbance eigenfunction ϕ satisfies the equation

$$\phi'' + \left(\frac{\rho'}{\rho}\right)\phi' - \left(\alpha^2 + \frac{U'' + \rho' U'/\rho}{U - \beta/\alpha}\right)\phi = 0 \tag{4}$$

where $(\,)'$ corresponds to $\mathrm{d}/\mathrm{d}y$. The equation just given reduces to the Rayleigh equation when the density is uniform. A "shooting" technique was used to solve this eigenvalue equation with the boundary condition

$$\phi(y \to \pm\infty) = e^{\mp \alpha y} \tag{5}$$

Equation (4) was integrated from both sides toward $y = 0$, and the matching of ϕ and ϕ' at this point yielded the spatial growth rate, $-\alpha_i$, of unstable disturbances and the corresponding wave number, α_r vs frequency β.

The flow patterns resulting from the amplification of the instability were determined from streakline calculations. The procedure for this calculation, which is essentially the same as that used by Michalke,[1] is outlined as follows. The perturbation velocities are given by

$$u(x,y,t) = \mathrm{real}\left(\frac{\partial \psi}{\partial y}\right), \qquad v(x,y,t) = \mathrm{real}\left(\frac{-\partial \psi}{\partial x}\right) \tag{6}$$

where ψ is given by Eq. (3) and ϕ is known from the solution of Eq. (4) for a specific set of eigenvalues (α, β). The motion of each fluid particle is then given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = U(y) + \epsilon u(x,y,t), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \epsilon v(x,y,t) \tag{7}$$

with the initial conditions $x(t=0) = x_0$ and $y(t=0) = y_0$. In Eq. (7), $U(y)$ is the undisturbed mean profile [Eq. (1)] and ϵ a measure of the initial magnitude of the disturbance. For the streaklines shown in the present work, x_0 was selected to be zero with $\epsilon = 0.0005$. Calculations were made by integrating Eq. (7), at different starting y locations, forward in time using the Euler method.

Results and Discussion

Uniform Density

The spatial instability characteristics of the shear layer with a wake component were calculated for a fixed velocity ratio as a function of the depth of the wake deficit (see the mean velocity profiles in Fig. 1). These profiles can be thought of as representing the evolution of the mean profile due to viscous diffusion. Starting near the splitter plate tip, a profile with a large wake deficit evolves into one with no wake component as the flow convects downstream. The main result, shown in Fig. 2, is that when the wake component is present, there are two unstable modes as opposed to one in the case of the hyperbolic tangent profile. Consistent with previous results,[5] as the wake deficit increases, the neutral point of mode 1, the stronger mode, moves to lower frequencies and its maximum amplification rate increases. We point out that the existence of mode 2, the weaker mode, had been known from the work of Miksad.[8] His results, however, were based on temporal stability calculations. Note from Fig. 2 that in the limit of the zero wake component, mode 1 approaches the tanh profile solution, while mode 2 vanishes. It should be mentioned that it is not clear whether the existence of two unstable modes is directly related to the presence of two inflection points in the mean velocity profile. We are not aware of any theoretical analysis that would predict the number of unstable modes based on the number of inflection points of the velocity profile.

The set of profiles in Fig. 3 was used to calculate the behavior of the solution in the limit of unity velocity ratio, namely, the case of pure wake. Results shown in Fig. 4 indicate that in this limit, the two modes of instability still persist. In the limit of pure wake, examination of the eigenfunctions (not shown here) revealed that modes 1 and 2 approach the "sinuous" and "varicose" modes, respectively, of wake instability (e.g., see the wake stability solutions of Mattingly and Criminale[9]).

Streaklines were calculated in order to examine the flow patterns that would result from the amplification of the unstable disturbance in each of the two modes (see the previous section for details). Calculations were performed only for the case of maximum amplification rate. The integration proceeded until the first structure rollup appeared. The results are illustrated in Figs. 5 and 6. We emphasize that the calculated streaklines are only intended to provide a qualitative description of the shear layer rollup patterns. Nonparallel flow and nonlinear effects are absent in these results.

The flow patterns in the case of the tanh profile, Fig. 5, are similar to those calculated by Michalke[1] (for $U_2 = 0$ shear layer) and show the familiar rollup of the shear layer into a vortex. Figure 6 depicts the streaklines for the two modes of instability

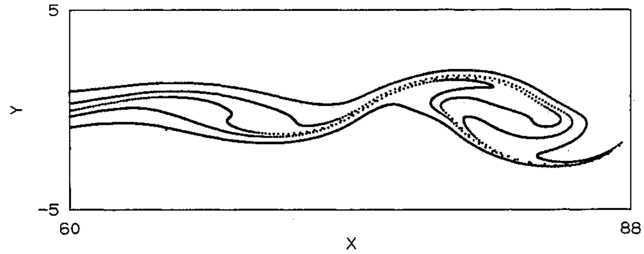


Fig. 5 Streaklines for uniform density tanh profile, $\lambda_u = 0.45$.

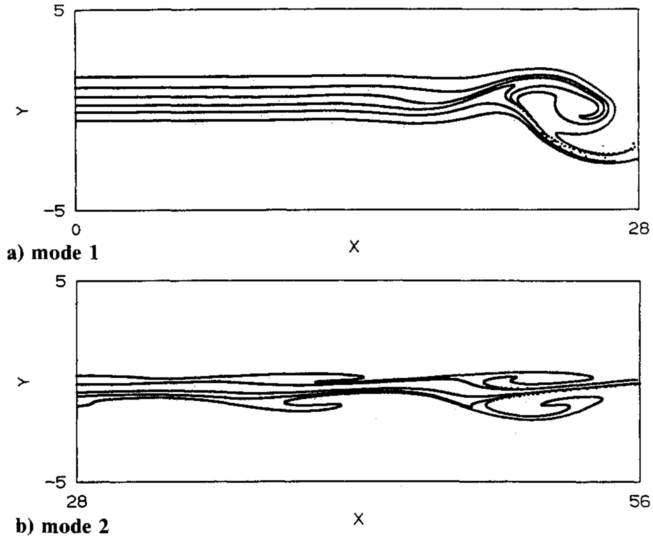


Fig. 6 Streaklines in uniform-density shear layer with wake component,; $\lambda_u = 0.45$, $W = 0.8$.

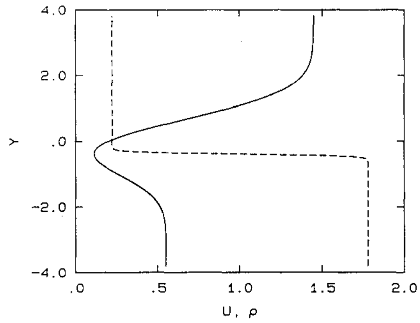


Fig. 7 Mean velocity and density profiles, $\lambda_u = 0.45$, $\lambda_\rho \approx -0.78$ ($\rho_2/\rho_1 = 8$), $1/\sigma = 16$.

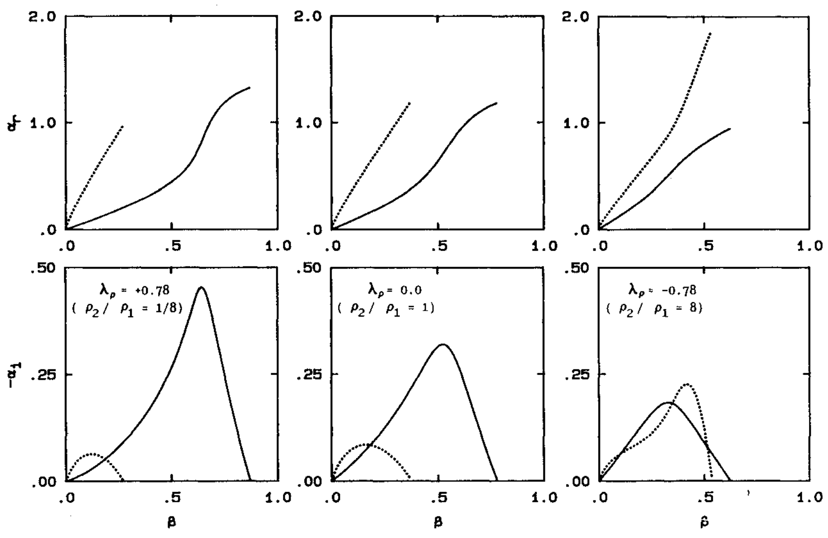
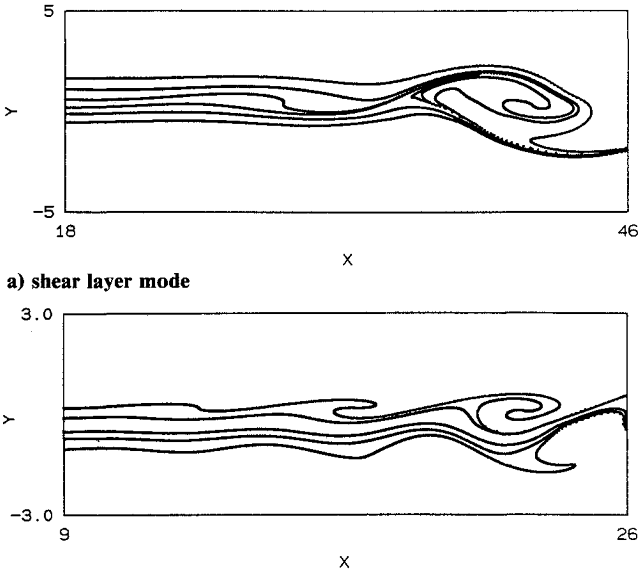


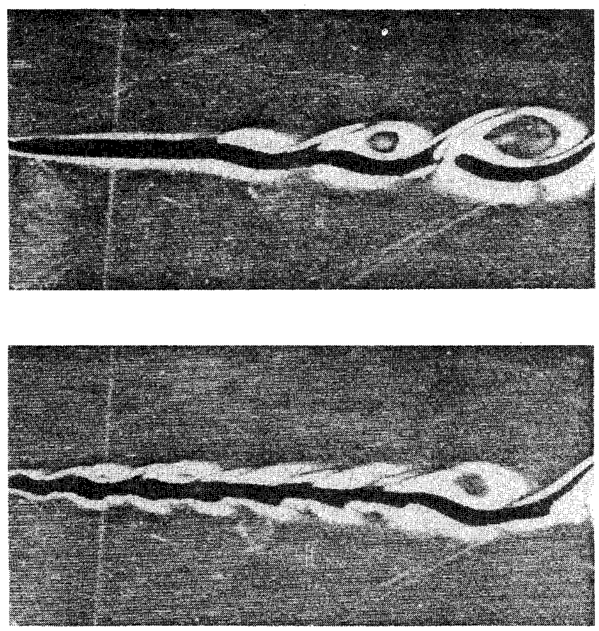
Fig. 8 Instability characteristics for nonuniform density. —— shear layer mode, · · · · wake mode.



a) shear layer mode

b) wake mode

Fig. 9 Streaklines in nonuniform density shear layer with wake component, $\lambda_\rho \approx -0.78$ ($\rho_2/\rho_1 = 8$).



a)

b)

Fig. 10 Schlieren photographs of the shear layer mode a) and the wake mode b). Flow is from left to right with high-speed stream on top ($U_2/U_1 \approx 0.38$, $\rho_2/\rho_1 \approx 10$).

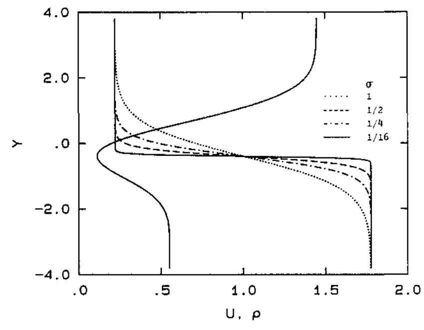


Fig. 11 Mean velocity and density profiles for different values of σ, $\lambda_u = 0.45$, $W = 0.8$, $\lambda_\rho \approx -0.78$ ($\rho_2/\rho_1 = 8$).

when the wake component is present. We note that the amplification of mode 1 leads to the usual Kelvin–Helmholtz-type shear layer rollup, whereas the mode 2 rollup patterns resemble a wake flow. We, therefore, hereafter refer to modes 1 and 2 as the "shear layer" and "wake" modes, respectively. The wake mode, in the uniform density case, is very difficult to observe experimentally since its amplification rate is much less than that of the shear layer mode through most of the unstable frequency range (see Fig. 2).

Nonuniform Density

In investigating the effect of nonuniform density, a specific velocity profile with $\lambda_u = 0.45$ ($U_2/U_1 \approx 0.38$) and a wake deficit of $W = 0.8$ was selected. We are particularly interested in the case of a low-speed stream having the higher density. The density profile was arranged to have its inflection point at the minimum of the velocity profile and its thickness much smaller [in Eq. (2), $1/\sigma = 16$] than that of the velocity profile; see Fig. 7. These conditions are expected to hold in the initial region of the flow near the splitter plate tip. The qualitative features of the results are not sensitive to these conditions as long as the density profile is "reasonably" thin relative to the velocity profile. See also the next section. Calculations were performed for $\rho_2/\rho_1 = 8$ ($\lambda_\rho \approx -0.78$). The case of high-density high-speed stream ($\rho_2/\rho_1 = 1/8$, $\lambda_\rho \approx +0.78$) was also calculated for comparison.

The most important result (see Fig. 8) is that when the high density is on the low-speed side, the two instability modes have similar amplification rates. In fact, the normally weak wake mode in the case of uniform density now has a slightly larger growth rate than the shear layer mode. The corresponding streaklines (at maximum amplification rate) for these two modes, Fig. 9, again illustrate the shear layer and wake-type rollup patterns. We also note from Fig. 8 that having the high density on the high-speed stream does not alter the relative significance of the two instability modes compared to the uniform-density shear layer.

The findings just described imply that, depending on the spectrum of the disturbances in the flow and the extent of the persistence of the wake component in the downstream region, a shear layer of nonuniform density may not rollup like the usual Kelvin-Helmholtz structure but more like a wake. These results also suggest that under the right flow conditions, both shear layer and wake modes of instability may exist simultaneously and interact with each other.

Since the shear layer and wake modes can be equally strong when the low-speed stream has the high-density fluid, an attempt was made to experimentally observe these modes. A shear layer ($U_2/U_1 \approx 0.38$) between a high-speed stream of helium and low-speed stream of argon ($\rho_2/U_1 \approx 10$) was forced acoustically. The flow visualization by schlieren photography, Fig. 10, shows that both shear layer and wake modes can be generated in a two-stream mixing layer. Note that the wake mode pattern, in Fig. 10b, appears to approach that of the shear layer mode toward the right side of photograph. This is to be expected, since as the flow moves downstream, the wake component of the velocity profile ultimately vanishes and only the shear layer instability mode remains.

Effect of the Density Profile Thickness

The case of uniform density can be considered to be equivalent to that of nonuniform density with a very large (in fact, infinite) thickness relative to the mean velocity profile. A question then remains as to how a weak wake mode in a uniform-density shear layer transforms into one equally strong as the shear layer mode when the high-density fluid is carried on the low-speed side. To shed some light on this question, the instability characteristics of the nonuniform density shear layer were calculated as a function of the density interface thickness [i.e., σ was varied in Eq. (2)]. The mean velocity and sample

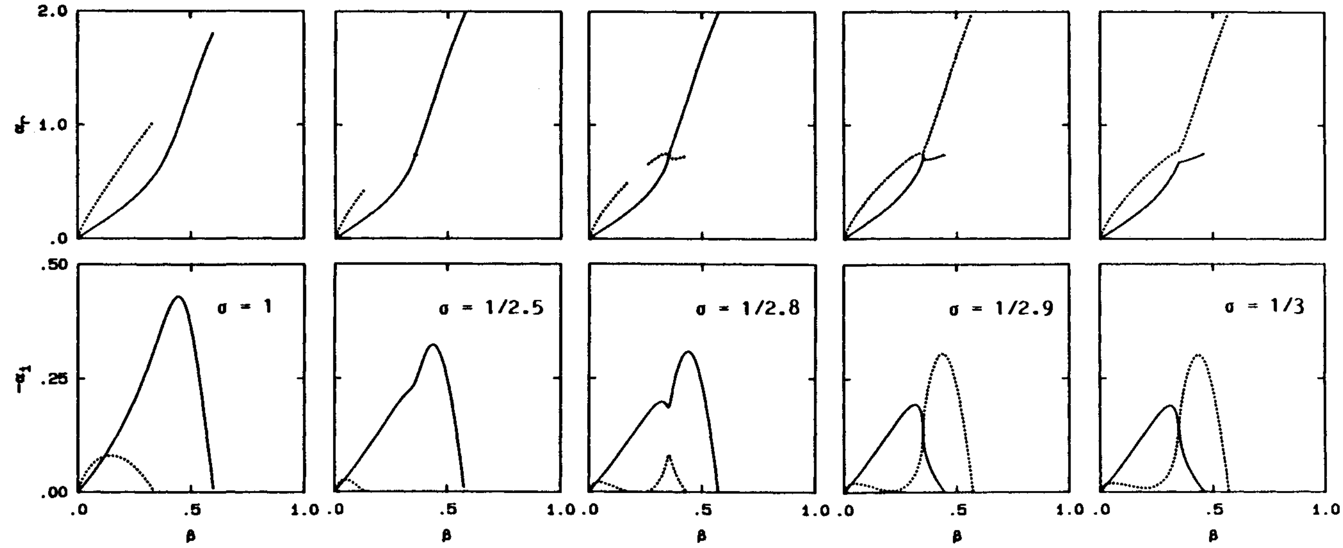


Fig. 12 Effect of the density profile thickness on the instability characteristics, $\lambda_u = 0.45$, $W = 0.8$, $\lambda_\rho \approx -0.78$ ($\rho_2/\rho_1 = 8$). —— shear layer mode, · · · · wake mode.

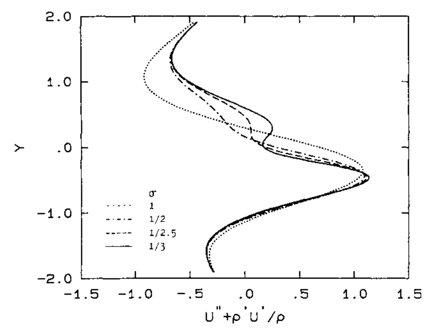


Fig. 13 Shape of ($U'' + \rho' U'/\rho$) profile for different density profile thicknesses.

density profiles are shown in Fig. 11. The results (see Fig. 12) illustrate that the density profile thickness must be smaller than a certain value before the wake mode becomes dominant. For these particular profiles, for example, it is required that the thickness of the density profile be at least 2.9 times smaller than the thickness of the velocity profile. On the other hand, when the high-density fluid is on the high-speed side, regardless of the density profile thickness (lowest value calculated was $\sigma = 1/16$), the shear layer mode of instability was found to be always dominant.

An interesting feature in Fig. 12 is that the appearance of a strong wake mode, between $\sigma = 1/2.8$ and $1/2.9$, seems to be a resonance phenomenon. This "repelling" of two otherwise identical eigenvalues is thought to be similar to the "Eckart" resonance,[10] which is the quantum-mechanical analog of the problem of two potential minima and the penetration of the potential barrier between them. In comparison with the Eckart resonance phenomenon, it seems that the behavior of the term ($U'' + \rho' U'/\rho$) in Eq. (4), in particular the presence of two maxima in its profile, is the determining factor. A plot of this term as a function of σ (see Fig. 13) shows the appearance of two maxima close to the value of σ where the resonance occurs (see Fig. 12).

Conclusions

The instability characteristics of uniform and nonuniform density plane shear layers were investigated. The mean velocity profile included a wake component in order to take account of the effect of the boundary layers on the two sides of the splitter plate. The range of unstable frequencies and wave numbers and the flow patterns resulting from the amplification of the instability were calculated using the inviscid, linear, parallel-flow, spatial stability analysis.

It was found that the shear layer with a wake component has two unstable modes. The growth of the unstable disturbance, in the shear layer mode, results in the usual Kelvin–Helmholtz rollup patterns, whereas in the wake mode, the flow patterns resemble a wake structure. When the density is uniform, the amplification rate of the wake mode is dominated by that of the shear layer mode. If the low-speed stream carries the high-density fluid, however, the two modes can become comparable in amplification rate. The opposite arrangement with the high-density fluid on the high-speed side behaves similarly to the uniform density case in that the shear layer mode remains the dominant mode. Schlieren flow-visualization pictures of a shear layer between a high-speed stream of a light gas (helium) and a low-speed stream of a heavy gas (argon) confirmed that both modes of instability could be excited experimentally. For the wake mode to become dominant, the thickness of the density profile, relative to the velocity profile thickness, must be smaller than a certain value. The onset of a strong wake mode, as the density profile thickness is reduced, appears to exhibit a resonance phenomenon. It is believed that this behavior is similar to the "Eckart" resonance.

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